A New Method for Frame Synchronization

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A new frame synchronization method based on an examination of the shortest bit length containing all error bursts is introduced. It is shown that the new method is more reliable and efficient than the commonly used one based on counting the number of errors for the convolutionally coded channel.

I. Introduction

Data from spacecraft are transmitted to stations on earth in encoded form. The transmission channels are generally noisy, and the encoding enables one to correct some of the errors. This is especially true of convolutional codes because of their high error correcting capability. The encoding/decoding procedure is briefly described in Fig. 1.

The encoded data appear in frames of fixed length and it is essential to identify the beginning of each frame. This is accomplished by the insertion of a marker at the beginning of each one. Correctly identifying the marker is the problem of frame synchronization. This is usually done after the received signal has passed through the Viterbi decoder. Several methods have been proposed for this purpose (see Ref. 1). Here we consider yet another method and make a comparative study of the new frame synchronization technique and the commonly used one based on counting the number of disagreements.

II. The Method

Data are transmitted in frames of N ($\cong$10080) bits, each of which begins with a marker of k(= 32) bits. Currently, frame synchronization involves counting the number of disagreements with the marker. Since Viterbi decoded data contains errors in bursts, we propose instead to examine the shortest length containing all the bursts. Specifically, choose a positive integer T (called threshold) and examine the k consecutive bits of data starting at a random point α. If the distance δ between the first and last disagreements with the marker in the sequence $\alpha, \alpha + 1, \ldots, \alpha + k - 1$ is greater than T, then we reject α as the beginning of a marker; otherwise it is retained as a candidate for a marker. In the latter event, we examine the k bits starting at the point $N + \alpha$. If $\delta > T$ for this sequence, we reject the k-bits starting at α (or $N + \alpha$) as a marker and repeat the procedure starting at $\alpha + 1$. Otherwise, we accept the k-bits as the first bit of a marker. In the latter case we continue to test the k-bits starting at $N + 2\alpha$, $N + 3\alpha, \ldots$ for the marker in the course of decoding data. If for three consecutive trials δ exceeds T then we reject α as the beginning of a marker, and repeat the procedure starting at $\alpha + 1$.

To analyze the performance of this technique we make the customary assumption that the $N - k$ bits of data in each frame is a random sequence of 0's and 1's, so that all sequences are equally likely to occur. Without loss of generality we may assume the marker consists of a sequence of k zeroes which due to the noise in the channel is possibly received erroneously. It has been observed that the errors in the Viterbi decoded data sequence occur in bursts and the lengths of these bursts follow the geometric distribution with parameter p. We recall that a bit sequence is a burst if (a) its first and last bits are incorrect, (b) it does not contain K (constraint length) consecutive correct bits, and (c) it is not contained in any other sequence with properties (a) and (b). Furthermore, the waiting time W between bursts has (shifted) geometric distribution with parameter q, i.e.,

$$P(W = n) = q(1-q)^{n-K+1} \quad (1)$$

Numerical values for p and q are given in Ref. 2.

Let M be the event that a marker actually starts at the randomly chosen point α, and O denote the event that α is identified as the beginning of a marker. In the next section we obtain estimates for $P(O|M)$, $P(M|O)$ and $P(O|M')$ where M' is the event complementary to M. These numbers provide a measure of the performance of the method. Note that $P(O'|M)$ and $P(O|M')$ are commonly called probability of miss and probability of false alarm respectively.

III. Probabilistic Estimates

To obtain estimates for the performance of the frame synchronization technique, it is convenient to assume that the probability of occurrence of more than one burst in a k-bit range is negligibly small. Such an assumption is reasonable for high SNR or short marker length. We first quantify this assertion.

Consider the sequence $1, 2, \ldots, k$, and let (a) $I = 1$, (b) $I = j$, $2 \leqslant j \leqslant k$ and (c) $I = \infty$ denote the events (a) 1 is contained in a burst, (b) first burst begins at bit j, and (c) there is no error in the sequence, respectively. Let L denote the length of the first burst where in case $I = 1$, L is measured from bit 1. Notice that if $I = j$ and $L = r$ then the bits $r + j$, $r + j + 1, \ldots, r + j + K - 1$ are correct. Denote by Y the starting point of the second burst. We want to calculate $P(Y \leqslant k)$. We have

$$Y = I + L + W$$

where W is the waiting time between bursts. Therefore

$$\begin{aligned} P(Y \leqslant k) &= \sum_{\substack{j+r \leqslant k-K \\ j \geqslant 1, r \geqslant 1}} P(Y \leqslant k \mid I = j, L = r)\ P(I = j, L = r) \\ &= \sum_{\substack{j+r \leqslant k-K \\ j \geqslant 1, r \geqslant 1}} P(W \leqslant k - j - r)\ P(I = j)\ P(L = r) \\ &= \sum_{\substack{j+r \leqslant k-K \\ j \geqslant 1, r \geqslant 1}} \left(\sum_{\nu = K-1}^{k-j-r} P(W = \nu) \right) P(I = j)\ P(L = r) \end{aligned} \quad (2)$$

From Eq. (1),

$$\sum_{\nu = K-1}^{k-j-r} P(W = \nu) = 1 - (1-q)^{k-j-r-K+2}$$

Also note

$$\left.\begin{aligned} P(I = 1) &= \frac{B}{B+W} \\ P(I = j) &= q(1-q)^{j-1}\ \theta \\ P(L = r) &= p(1-p)^{r-1} \end{aligned}\right\} \quad (3)$$

where

$$\theta = \frac{W}{B+W} \quad \text{for} \quad 2 \leqslant j \leqslant k$$

Therefore

$$\begin{aligned} P(Y \leqslant k) = & \sum_{\substack{r \geqslant 1 \\ r \leqslant k-K-1}} \frac{B}{B+W}\ p(1-p)^{r-1} \\ & - \sum_{\substack{r \geqslant 1 \\ r \leqslant k-K-1}} \frac{B}{B+W}\ p(1-p)^{r-1}\ (1-q)^{k-r+K+1} \\ & + \sum_{\substack{j \geqslant 2,\ r \geqslant 1 \\ j+r \leqslant k-K}} \theta\ pq\ (1-p)^{r-1}\ (1-q)^{j-1} \\ & - \sum_{\substack{j \geqslant 2,\ r \geqslant 1 \\ j+r\ k-K}} \theta\ pq\ (1-p)^{r-1}\ (1-q)^{k-r+K+1} \end{aligned}$$

i.e.,

$$P(Y \leqslant k) = \frac{B}{B+W}\left(1-(1-p)^{k-K-1}\right)$$

$$-\frac{B}{B+W}\ \frac{p}{p-q}\ (1-q)^{2+2K}$$

$$\cdot\left((1-q)^{k-K-1}-(1-p)^{k-K-1}\right)$$

$$+\sum_{j\geqslant 2,\, r\geqslant 1} \frac{W}{B+W}\, pq\,(1-p)^r\,(1-q)^{j-1}$$

$$-\frac{W}{B+W}\, pq\,(1-q)^{k+K} \sum_{r=1}^{k-K}\left(\frac{1-p}{1-q}\right)^{r-1}(k-K-r-1) \tag{4}$$

The numerical values of $P(Y \leqslant k)$ for different values of SNR are given in Table 1. The parameters $p = 1/B$ and $q = 1/(W\text{-}5)$ are taken from table C-1 in Ref. 2.

To estimate $P(M \mid O)$ we first note

$$P(M \mid O) = \frac{P(M, O)}{P(O)}$$

$$= \frac{P(O \mid M)\ P(M)}{P(O \mid M)P(M) + P(O \mid M')P(M')} \tag{5}$$

Let C be the event that the first burst point occurs no sooner than $K - T + 1$. Then it is trivial that

$$P(\widetilde{O} \mid M) \geqslant P(C) + P(1 \leqslant B \leqslant k - T)\, P(L \leqslant T)$$

where $\widetilde{O}$ denotes the event that $\delta \leqslant T$ for the sequence α, $\alpha + 1, \ldots, \alpha + k - 1$ where α is the randomly chosen starting bit. We assume, without loss of generality, that bit 1 is the beginning of a marker. Now

$$P(C) = \theta \sum_{j=k-T+1}^{\infty} q(1-q)^{j-1}$$

$$= \theta\,(1-q)^{k-T}$$

and

$$P(1 \leqslant B \leqslant k - T) = 1 - \theta\,(1-q)^{k-T}$$

(For a justification of using parameter q in evaluation of $P(C)$, see Ref. 3, pp. 12-13.) Hence

$$P(\widetilde{O} \mid M) \geqslant (1-q)^{k-T} + \left(1 - \theta\,(1-q)^{k-T}\right) \cdot \left(1 - (1-p)^T\right)$$

We denote r.h.s. of the above inequality by β. Since separation $N - k$ between markers is sufficiently large, the error bursts in the sequences beginning at α and $N + \alpha$ are essentially independent. Therefore

$$P(O \mid M) = P(\widetilde{O} \mid M)^{\ 2} \geqslant \beta^2 \tag{6}$$

Since for $0 < a < 1$, $c > 0$ the function

$$q(x) = ax/ax + (1-a)c$$

is increasing, we obtain from Eq. (5)

$$P(M \mid O) \geqslant \frac{\beta^2\, P(M)}{\beta^2\, P(M) + P(O \mid M')\, P(M')} \tag{7}$$

To calculate $P(O \mid M')$ we use the acceptability assumption on the marker. Thus if the randomly chosen point α is such that the sequence $\{\alpha, \alpha + 1, \ldots, \alpha + k - 1\}$ overlaps with but is not identical with the marker, then the probability of retaining α as the beginning of a marker is no greater than the case where the marker does not overlap. Now if $\{\alpha, \ldots, \alpha + k - 1\}$ does not overlap with the marker then this probability is bounded by

$$\gamma = \frac{\lambda(k, T)}{2^k}$$

where $\lambda(k, T)$ is one plus the number of binary sequences $a(i), a(i+1), \ldots, a(i + T - 1)$ such that $a(i) = 1$, $1 \leqslant i$, $i + T - 1 \leqslant k$. We have

$$\lambda(k, T) = 1 + k + (k-1) + (k-2) + \ldots + (k - T + 1)\, 2^{T-2}$$

$$= k + k\,(2^{T-1} - 1) - (T-1)\, 2^T + T\, 2^{T-1}$$

Substituting

$$P(O \mid M') \leqslant \gamma^2 \tag{8}$$

in Eq. (7) we get

$$P(M \mid O) \geqslant \frac{\beta^2}{\beta^2 + (N-1)\,\gamma^2} \tag{9}$$

The above inequality and Eqs. (6) and (8) are the required estimates.

IV. Numerical Calculations

In this section we make a numerical comparison of the quantities $P(M|O)$, $P(O|M')$ and $P(O|M)$ at various SNR values for the new method and one based on counting the number of disagreements. A few remarks are necessary regarding our calculations.

(1) To evaluate $P(O|M)$ and $P(O|M')$ for the old method we use Eq. (5), and calculate $P(O|M)$ and $P(O|M')$ by simulation.

(2) More precisely, we used a random number generator to generate bursts in 32,000 markers at various SNR values, and $P(O|M)$ was calculated accordingly.

(3) For calculation of $P(O|M')$ we assumed the marker is acceptable in the sense of Ref. 1. Thus if the randomly chosen point α is such that the sequence $\{\alpha, \alpha + 1, \ldots, \alpha + k - 1\}$ overlaps with but is not identical with the marker, we still can treat it as one in random data.

(4) Notice that our calculations sometimes give only upper or lower bounds.

(5) The quantity $P(O|M')$ is independent of SNR since we are assuming randomness of data.

(6) To make a meaningful comparison of the two methods we have graphically exhibited $P(O|M)$ and $P(M|O)$ for the same values of $P(O|M')$ (Figs. 2–8). For $P(M|O)$ we have only exhibited the curves for SNR = 1.6 since $P(M|O)$ is very stable relative to the variation of SNR.

(7) Detailed results of our calculations appear in Tables 1–14 and Figs. 2–8 below.

V. Conclusion

A comparison of performance statistics for fixed probability of false detection shows that the new method is significantly more reliable and efficient in detection of the marker. Mathematically, false detection, reliability and efficiency in detection of the marker are measured by $P(O|M')$, $P(M|O)$ and $P(O|M)$, respectively. The reliability of both methods is very stable relative to the variation of the signal to noise ratio.

References

1. Swanson, L., 1982, *A Comparison Frame Synchronization Methods*, JPL Publication 82-100, Jet Propulsion Laboratory, Pasadena, Calif.

2. Miller, R. L., L. J. Deutsch and S. A. Butman, 1981, *On the Error Statistics of Viterbi Decoding and the Performance of Concatenated Codes*, JPL Publication 81-9, Jet Propulsion Laboratory, Pasadena, Calif.

3. Feller, W., 1971. *An Introduction to Probability Theory and its Applications*, vol. 2. New York: John Wiley.

Table 1. Probability of more than one burst

SNR	$P\,(Y \leqslant 32)$
3	9×10^{-6}
2.5	8×10^{-5}
2.1	4×10^{-4}
1.9	10^{-3}
1.8	10^{-3}
1.7	2×10^{-3}
1.6	3×10^{-3}

Table 2. Comparison of reliability, SNR = 1.6

Threshold (T)	New Method $P(M \mid O) \geqslant$	Old Method $P(M \mid O)$
1	0.999+	0.999+
2		0.999
3		0.986
4		0.904
5		0.620
6		0.257
7		0.082
8		0.026
9		--
10		--
11		--
12		--
13		--
14		--
15		--
16		--
17	0.999+	--
18	0.997	--
19	0.991	--
20	0.971	--
21	0.906	--

Table 3. Comparison of reliability, SNR = 1.7

Threshold (T)	New Method $P(M \mid O) \geqslant$	Old Method $P(M \mid O)$
1	0.999+	0.999+
2		0.999
3		0.986
4		0.906
5		0.623
6		0.261
7		0.083
8		0.027
9		--
10		--
11		--
12		--
13		--
14		--
15		--
16		--
17	0.999+	--
18	0.998	--
19	0.991	--
20	0.971	--
21	0.906	--

Table 4. Comparison of reliability, SNR = 1.8

Threshold (T)	New Method $P(M\mid O) \geqslant$	Old Method $P(M\mid O)$
1	0.999+	0.999+
2		0.999
3		0.986
4		0.907
5		0.626
6		0.263
7		0.084
8		0.027
9		--
10		--
11		--
12		--
13		--
14		--
15		--
16		--
17	0.999+	--
18	0.998	--
19	0.991	--
20	0.971	--
21	0.907	--

Table 5. Comparison of reliability, SNR = 1.9

Threshold (T)	New Method $P(M\mid O) \geqslant$	Old Method $P(M\mid O)$
1	0.999+	0.999+
2		0.999
3		0.987
4		0.908
5		0.628
6		0.264
7		0.082
8		0.027
9		--
10		--
11		--
12		--
13		--
14		--
15		--
16		--
17	0.999+	--
18	0.998	--
19	0.992	--
20	0.971	--
21	0.906	--

Table 6. Comparison of efficiency, SNR = 2.1

Threshold (T)	New Method $P(M \mid O) \geqslant$	Old Method $P(M \mid O)$
1	0.999+	0.999+
2		0.999
3		0.987
4		0.909
5		0.632
6		0.267
7		0.085
8		0.027
9		--
10		--
11		--
12		--
13		--
14		--
15		--
16		--
17	0.999+	--
18	0.998	--
19	0.992	--
20	0.971	--
21	0.907	--

Table 7. Comparison of reliability, SNR = 2.5

Threshold (T)	New Method $P(M \mid O) \geqslant$	Old Method $P(M \mid O)$
1	0.999+	0.999+
2		0.999
3		0.987
4		0.910
5		0.635
6		0.269
7		0.086
8		0.027
9		--
10		--
11		--
12		--
13		--
14		--
15		--
16		--
17	0.999+	--
18	0.998	--
19	0.992	--
20	0.971	--
21	0.908	--

Table 8. Comparison of efficiency, SNR = 1.6

Threshold (T)	New Method $P(M\mid O) \geqslant$	Old Method $P(M\mid O)$
1	0.864	0.878
2	0.875	0.894
3	0.885	0.908
4	0.895	0.920
5	0.903	0.930
6	0.911	0.941
7	0.919	0.949
8	0.926	0.957
9	0.932	--
10	0.938	--
11	0.943	--
12	0.948	--
13	0.953	--
14	0.957	--
15	0.961	--
16	0.964	--
17	0.968	--
18	0.971	--
19	0.973	--
20	0.976	--
21	0.978	--

Table 9. Comparison of efficiency, SNR = 1.7

Threshold (T)	New Method $P(M\mid O) \geqslant$	Old Method $P(M\mid O)$
1	0.884	0.897
2	0.894	0.911
3	0.903	0.922
4	0.911	0.933
5	0.919	0.942
6	0.926	0.950
7	0.932	0.958
8	0.938	0.965
9	0.943	--
10	0.948	--
11	0.953	--
12	0.957	--
13	0.961	--
14	0.965	--
15	0.968	--
16	0.971	--
17	0.974	--
18	0.976	--
19	0.979	--
20	0.981	--
21	0.983	--

Table 10. Comparison of efficiency, SNR = 1.8

Threshold (T)	New Method $P(M\mid O) \geqslant$	Old Method $P(M\mid O)$
1	0.905	0.915
2	0.914	0.927
3	0.921	0.937
4	0.928	0.946
5	0.935	0.955
6	0.941	0.962
7	0.946	0.968
8	0.951	0.974
9	0.956	--
10	0.960	--
11	0.963	--
12	0.967	--
13	0.970	--
14	0.973	--
15	0.976	--
16	0.978	--
17	0.980	--
18	0.982	--
19	0.984	--
20	0.986	--
21	0.987	--

Table 11. Comparison of efficiency, SNR = 1.9

Threshold (T)	New Method $P(M\mid O) \geqslant$	Old Method $P(M\mid O)$
1	0.915	0.929
2	0.923	0.939
3	0.930	0.948
4	0.937	0.956
5	0.943	0.962
6	0.948	0.968
7	0.953	0.974
8	0.958	0.978
9	0.962	--
10	0.966	--
11	0.969	--
12	0.972	--
13	0.975	--
14	0.977	--
15	0.980	--
16	0.982	--
17	0.984	--
18	0.986	--
19	0.987	--
20	0.988	--
21	0.990	--

Table 12. Comparison of efficiency, SNR = 2.1

Threshold (T)	New Method $P(M \mid O) \geqslant$	Old Method $P(M \mid O)$
1	0.948	0.955
2	0.954	0.961
3	0.958	0.967
4	0.962	0.973
5	0.966	0.977
6	0.969	0.981
7	0.972	0.984
8	0.975	0.987
9	0.978	--
10	0.980	--
11	0.982	--
12	0.984	--
13	0.986	--
14	0.987	--
15	0.989	--
16	0.990	--
17	0.991	--
18	0.992	--
19	0.993	--
20	0.994	--
21	0.994	--

Table 13. Comparison of efficiency, SNR = 2.5

Threshold (T)	New Method $P(M \mid O) \geqslant$	Old Method $P(M \mid O)$
1	0.977	0.981
2	0.980	0.986
3	0.983	0.987
4	0.985	0.989
5	0.987	0.990
6	0.988	0.992
7	0.990	0.994
8	0.991	0.996
9	0.992	--
10	0.993	--
11	0.994	--
12	0.995	--
13	0.995	--
14	0.996	--
15	0.996	--
16	0.997	--
17	0.997	--
18	0.998	--
19	0.998	--
20	0.998	--
21	0.999	--

Table 14. Probability of false detection

Threshold (T)	New Method $P(O\|M') \leqslant$	Old Method $P(O\|M')$
1	10^{-16}	10^{-16}
2	10^{-16}	10^{-14}
3	10^{-15}	10^{-12}
4	10^{-15}	10^{-10}
5	10^{-14}	10^{-9}
6	10^{-14}	10^{-7}
7	10^{-13}	10^{-6}
8	10^{-12}	10^{-5}
9	10^{-12}	--
10	10^{-11}	--
11	10^{-11}	--
12	10^{-10}	--
13	10^{-10}	--
14	10^{-9}	--
15	10^{-8}	--
16	10^{-8}	--
17	10^{-7}	--
18	10^{-7}	--
19	10^{-6}	--
20	10^{-6}	--
21	10^{-5}	--

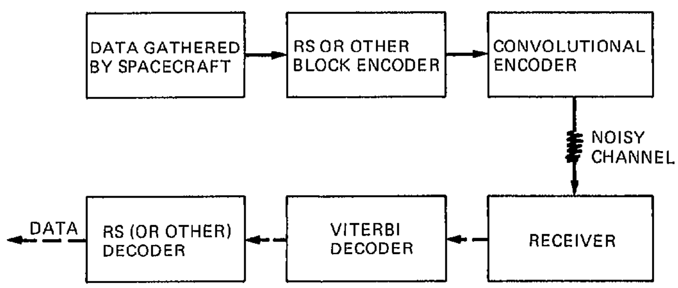

Fig. 1. The encoding-decoding procedure

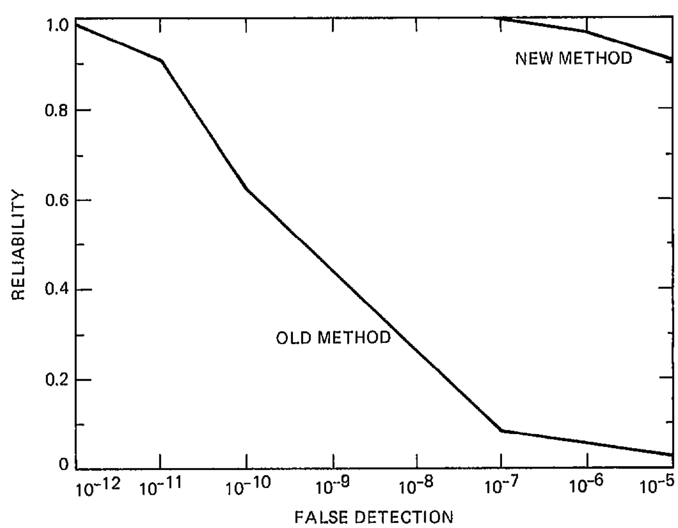

Fig. 2. Graph 1

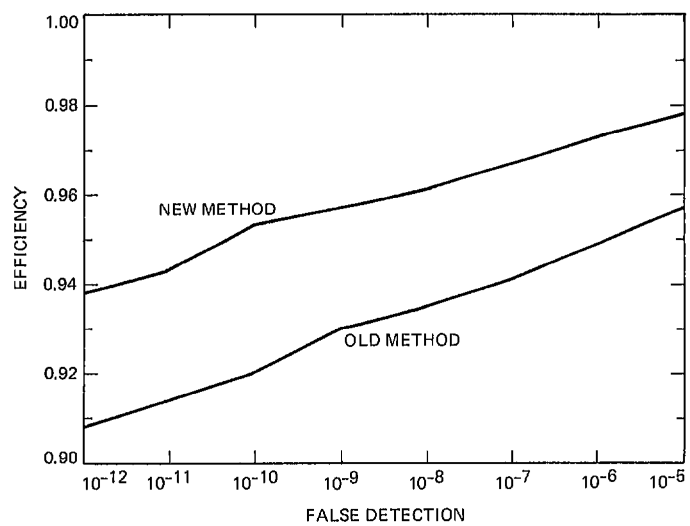

Fig. 3. Graph 2, SNR = 1.6

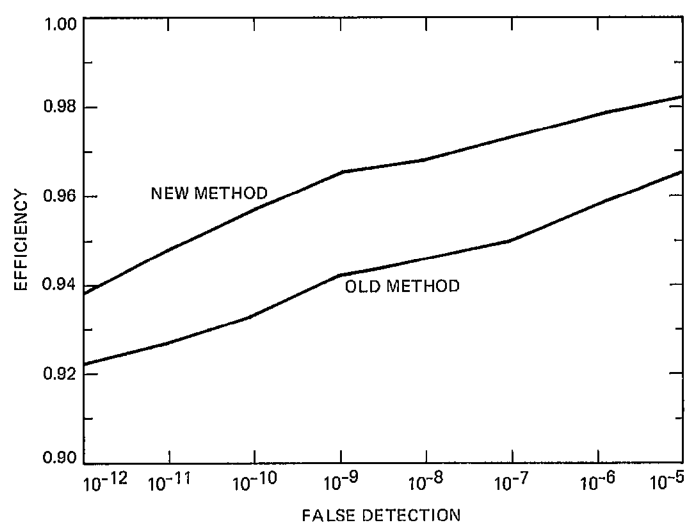

Fig. 4. Graph 3, SNR = 1.7

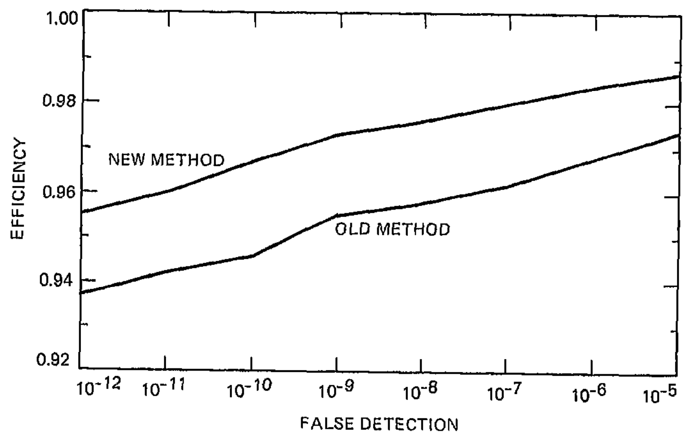

Fig. 5. Graph 4, SNR = 1.8

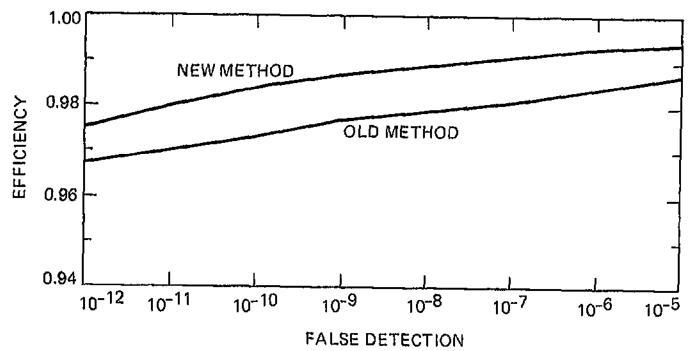

Fig. 7. Graph 6, SNR = 2.1

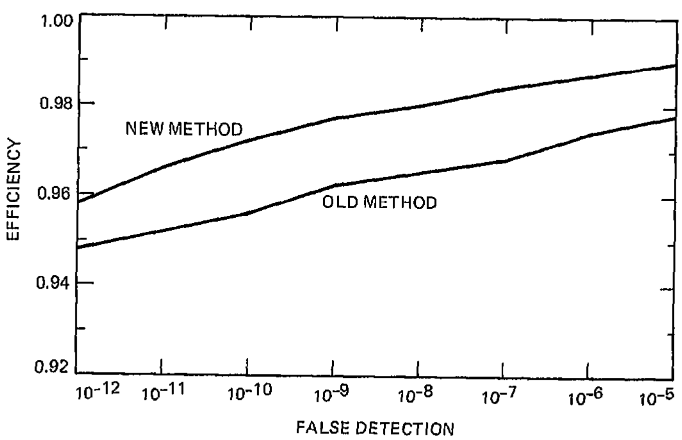

Fig. 6. Graph 5, SNR = 1.9

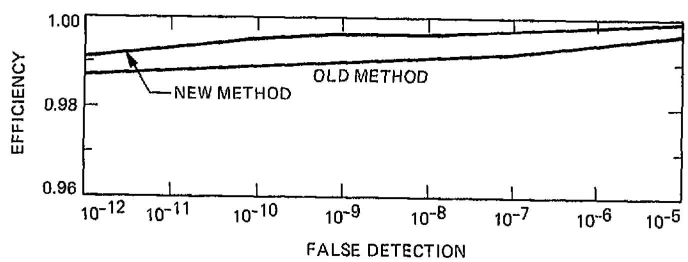

Fig. 8. Graph 7, SNR = 2.5